

OPERATOR ACZEL INEQUALITY

MOHAMMAD SAL MOSLEHIAN

ABSTRACT. We establish several operator versions of the classical Aczel inequality. One of operator versions deals with the weighted operator geometric mean and another is related to the positive sesquilinear forms. Some applications including the unital positive linear maps on C^* -algebras and the unitarily invariant norms on matrices are presented.

1. INTRODUCTION

Let $\mathbb{B}(\mathcal{H})$ denote the algebra of all bounded linear operators acting on a complex Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ and I is the identity operator. In the case where $\dim \mathcal{H} = n$, we identify $\mathbb{B}(\mathcal{H})$ with the full matrix algebra $M_n(\mathbb{C})$ of all $n \times n$ matrices with entries in the complex field \mathbb{C} . An operator $A \in \mathbb{B}(\mathcal{H})$ is called positive (positive-semidefinite for matrices) if $\langle A\xi, \xi \rangle \geq 0$ holds for every $\xi \in \mathcal{H}$ and then we write $A \geq 0$. For $A, B \in \mathbb{B}(\mathcal{H})$, we say $A \leq B$ if $B - A \geq 0$. Let f be a continuous real valued function defined on an interval J . The function f is called operator decreasing if $B \leq A$ implies $f(A) \leq f(B)$ for all A, B with spectra in J . A function f is said to be operator concave on J if

$$\lambda f(A) + (1 - \lambda)f(B) \leq f(\lambda A + (1 - \lambda)B)$$

for all $A, B \in \mathbb{B}(\mathcal{H})$ with spectra in J and all $\lambda \in [0, 1]$.

By a C^* -algebra we mean a closed $*$ -subalgebra \mathcal{A} of $\mathbb{B}(\mathcal{H})$ for some Hilbert space \mathcal{H} . Any finite dimensional C^* -algebra is isometrically $*$ -isomorphic to a direct sum of finitely many full matrix algebras. A C^* -algebra is called unital if it has an identity. A map taking the identity to identity is called unital. A map $\Phi : \mathcal{A} \rightarrow \mathcal{B}$ between C^* -algebras is called positive if it takes positive operators to positive ones, in particular, a positive linear map from \mathcal{A} into \mathbb{C} is called a positive linear functional. A map $\Phi : \mathcal{A} \rightarrow \mathcal{B}$ is called 2-positive if the map $\Phi_2 : M_2(\mathcal{A}) \rightarrow M_2(\mathcal{B})$ defined by $\Phi_2([a_{ij}]) = [\Phi(a_{ij})]$ is positive, where $M_2(\mathcal{A})$

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is the C^* -algebra all 2×2 matrices with entries in \mathcal{A} . An operator A is called a contraction if $\|A\| \leq 1$. We refer the reader to [11] for undefined notions on operator theory and to [8] for more information on operator inequalities.

In 1956, Aczél [3] proved that if a_i, b_i ($1 \leq i \leq n$) are positive real numbers such that $a_1^2 - \sum_{i=2}^n a_i^2 > 0$ or $b_1^2 - \sum_{i=2}^n b_i^2 > 0$, then

$$\left(a_1 b_1 - \sum_{i=2}^n a_i b_i\right)^2 \geq \left(a_1^2 - \sum_{i=2}^n a_i^2\right) \left(b_1^2 - \sum_{i=2}^n b_i^2\right).$$

The Aczél inequality has some applications in mathematical analysis and in the theory of functional equations in non-Euclidean geometry. During the last decades several interesting generalization of this significant inequality were obtained. Popoviciu [12] extended Aczél's inequality by showing that

$$\left(a_1 b_1 - \sum_{i=2}^n a_i b_i\right)^p \geq \left(a_1^p - \sum_{i=2}^n a_i^p\right) \left(b_1^p - \sum_{i=2}^n b_i^p\right),$$

if $p \geq 1$ and $a_1^p - \sum_{i=2}^n a_i^p > 0$ or $b_1^p - \sum_{i=2}^n b_i^p > 0$. Aczél's inequality and Popoviciu's inequality was sharpened by Wu [15], see also [14]. A variant of Aczél's inequality in inner product spaces was given by Dragomir [5] by establishing that if a, b are real numbers and x, y are vectors of an inner product space such that $a^2 - \|x\|^2 > 0$ or $b^2 - \|y\|^2 > 0$, then $(a^2 - \|x\|^2)(b^2 - \|y\|^2) \leq (ab - \operatorname{Re}\langle x, y \rangle)^2$, see also [6]. Cho, Matić and Pečarić [4] generalized Aczél's inequality for linear isotonic functionals and convex functions. Several Aczél type inequalities involving norms in Banach spaces were presented by Mercer [9]. Also, Sun [13] gave an Aczél–Chebyshev type inequality for positive linear functionals. To find operator versions of Hua's inequality (see [10]), Fujii [7, Theorem 3] obtained an Aczél operator inequality by showing that if $\Phi : \mathcal{A} \rightarrow \mathcal{B}$ is a contractive 2-positive unital linear map between unital C^* -algebras, $A, B \in \mathcal{A}$ are contraction and $\Phi(B^*A)$ is normal with the polar decomposition $\Phi(B^*A) = U|\Phi(B^*A)|$, then

$$|1 - \Phi(B^*A)| \geq (1 - \Phi(A^*A))\sharp U^*(1 - \Phi(B^*B))U.$$

In this paper we establish several operator versions of the classical Aczel inequality. One of operator versions deals with the weighted operator geometric mean $A\sharp_t B := A^{1/2}(A^{-1/2}BA^{-1/2})^t A^{1/2}$ ($t \in [0, 1]$) and another is related to the positive sesquilinear forms. Some applications including the unital positive linear maps on C^* -algebras and the unitarily invariant norms on matrices are presented. Recall that a unitarily invariant norm $||| \cdot |||$ has the property $|||UXV||| = |||X|||$, where

U and V are unitaries and $X \in M_n(\mathbb{C})$. For more information on the theory of the unitarily invariant norms the reader is referred to [1].

2. OPERATOR ACZEL INEQUALITY VIA GEOMETRIC MEAN

We start this section with a lemma about a parameterized operator power mean m_t satisfying $Am_tB \leq (1-t)A + tB$ for any two positive invertible operators A, B . The power means $A\sharp_{r,t}B := A^{1/2}(1-t+t(A^{-1/2}BA^{-1/2})^r)^{1/r}A^{1/2}$ ($r \in [-1, 1] \setminus \{0\}$) and $A\sharp_{0,t}B := A\sharp_tB$ ($t \in [0, 1]$) are such parameterized operator power means. Clearly if $AB = BA$, then $A\sharp_tB = A^{1-t}B^t$; see [8, Chapter V].

Lemma 2.1. *Suppose that m_t is a parameterized operator power mean not greater than the weighted arithmetic mean. If J is an interval of $(0, \infty)$ and $f : J \rightarrow (0, \infty)$ is operator decreasing and operator concave on J and $A, B \in \mathbb{B}(\mathcal{H})$ are positive invertible operators with spectra contained in J , then*

$$f(Am_tB) \geq f(A)m_tf(B) \quad (2.1)$$

Proof. It follows from $Am_tB \leq (1-t)A + tB$ that

$$\begin{aligned} f(Am_tB) &\geq f((1-t)A + tB) && \text{(since } f \text{ is operator decreasing)} \\ &\geq (1-t)f(A) + tf(B) && \text{(since } f \text{ is operator concave)} \\ &\geq f(A)m_tf(B) && \text{(by the property of } m_t \text{).} \end{aligned} \quad (2.2)$$

□

Theorem 2.2. *Let J be an interval of $(0, \infty)$, let $f : J \rightarrow (0, \infty)$ be operator decreasing and operator concave on J , $\frac{1}{p} + \frac{1}{q} = 1$, $p, q > 1$ and let $A, B \in \mathbb{B}(\mathcal{H})$ be positive invertible operators with spectra contained in J . Then*

$$f(A\sharp_{1/q}^p B^q) \geq f(A^p)\sharp_{1/q}f(B^q) \quad (2.3)$$

$$\langle f(A\sharp_{1/q}^p B^q)\xi, \xi \rangle \geq \langle f(A^p)\xi, \xi \rangle^{\frac{1}{p}} \langle f(B^q)\xi, \xi \rangle^{\frac{1}{q}}. \quad (2.4)$$

for any vector $\xi \in \mathcal{H}$.

Proof. Lemma 2.1 yields inequality (2.4).

Let $\xi \in \mathcal{H}$ be an arbitrary vector. It follows from (2.2) that

$$\begin{aligned} \langle f(A\sharp_{1/q}^p B^q)\xi, \xi \rangle &\geq \frac{1}{p} \langle f(A^p)\xi, \xi \rangle + \frac{1}{q} \langle f(B^q)\xi, \xi \rangle \\ &\geq \langle f(A^p)\xi, \xi \rangle^{\frac{1}{p}} \langle f(B^q)\xi, \xi \rangle^{\frac{1}{q}} \end{aligned}$$

(weighted arithmetic-geometric mean inequality).

□

Remark 2.3. The Hölder–McCarthy inequality asserts that if $C \in \mathbb{B}(\mathcal{H})$ is a positive operator, then $\langle C^r \xi, \xi \rangle \leq \langle C \xi, \xi \rangle^r$ for all $0 < r < 1$ and all unit vectors $\xi \in \mathcal{H}$; cf. [8, Theorem 1.4]. It follows from (2.4) that

$$\langle f(A^p \sharp_{1/q} B^q) \xi, \xi \rangle \geq \langle f(A^p)^{\frac{1}{p}} \xi, \xi \rangle \langle f(B^q)^{\frac{1}{q}} \xi, \xi \rangle.$$

Thus

$$\|f(A^p \sharp_{1/q} B^q)^{1/2} \xi\| \geq \|f(A^p)^{\frac{1}{2p}} \xi\| \|f(B^q)^{\frac{1}{2q}} \xi\|.$$

Corollary 2.4. *Let $\frac{1}{p} + \frac{1}{q} = 1$, $p, q > 1$ and $A, B \in \mathbb{B}(\mathcal{H})$ be commuting positive invertible operators with spectra contained in $(0, 1)$. Then*

$$1 - \|AB\xi\|^2 \geq \langle (1 - \|A^{p/2}\xi\|^2)^{\frac{1}{p}} (1 - \|B^{q/2}\xi\|^2)^{\frac{1}{q}} \rangle.$$

for any unit vector $\xi \in \mathcal{H}$.

Proof. Apply Theorem 2.2 to the function $f(t) = 1 - t$ on $(0, 1)$ and note that $A^p \sharp_{1/q} B^q = AB$. □

Corollary 2.5. *Let J be an interval of $(0, \infty)$, let $f : J \rightarrow (0, \infty)$ be operator decreasing and operator concave on J and $A, B \in \mathbb{B}(\mathcal{H})$ be positive invertible commuting operators with spectra contained in J . Then*

$$f(AB) \geq (f(A^p))^{1/p} (f(B^q))^{1/q}.$$

Corollary 2.6. *If f is a decreasing concave function on an interval J and a_i, b_i ($1 \leq i \leq n$) are positive numbers in J , then*

$$\sum_{i=1}^n f(a_i b_i) \geq \left(\sum_{i=1}^n f(a_i^p) \right)^{1/p} \left(\sum_{i=1}^n f(b_i^q) \right)^{1/q}.$$

Apply (2.4) to the positive operators $A(x_1, \dots, x_n) = (a_1 x_1, \dots, a_n x_n)$ and $B(x_1, \dots, x_n) = (b_1 x_1, \dots, b_n x_n)$ acting on the Hilbert space $\mathcal{H} = \mathbb{C}^n$ and $\xi = (1, 1, \dots, 1)$.

3. ACZEL'S INEQUALITY VIA POSITIVE LINEAR FUNCTIONALS

In this section we present a new version of Aczel's inequality through positive linear functionals. The first result is a generalization of the main result of [5]. The proof differs from that the main result of [5]. It is an Aczél type inequality for sesquilinear forms.

Theorem 3.1. *Let $\phi(.,.)$ be a positive sesquilinear form on a linear space \mathcal{X} , let $x, y \in \mathcal{X}$ such that $\phi(x, x) \leq M_1^2$ or $\phi(y, y) \leq M_2^2$ for some positive numbers M_1, M_2 and let $L : \mathbb{C} \rightarrow \mathbb{R}$ be a function fulfilling $L(z) \leq |z|$ for all $z \in \mathbb{C}$. Then*

$$(M_1 M_2 - L(\phi(x, y)))^2 \geq (M_1^2 - \phi(x, x))(M_2^2 - \phi(y, y)).$$

Proof. We may assume that both $\phi(x, x) \leq M_1^2$ and $\phi(y, y) \leq M_2^2$ hold. Then the Cauchy–Schwarz inequality implies that $|\phi(x, y)| \leq M_1 M_2$. We have

$$\begin{aligned} (M_1 M_2 - L(\phi(x, y)))^2 &\geq (M_1 M_2 - |\phi(x, y)|)^2 \\ &\geq \left(M_1 M_2 - \sqrt{\phi(x, x)\phi(y, y)} \right)^2 \quad (\text{Cauchy–Schwarz inequality}) \\ &\geq \left(M_1 M_2 - \frac{\phi(x, x) + \phi(y, y)}{2} \right)^2 \quad (\text{arithmetic-geometric mean ineq.}) \\ &\geq \left(\frac{M_1^2 + M_2^2 - \phi(x, x) - \phi(y, y)}{2} \right)^2 \\ &\geq (M_1^2 - \phi(x, x))(M_2^2 - \phi(y, y)) \quad (\text{arithmetic-geometric mean ineq.}) \end{aligned}$$

as desired. \square

The next result is a consequence of Theorem 3.1, but we present a different proof for it.

Theorem 3.2. *Suppose that $\Phi : \mathcal{A} \rightarrow \mathcal{B}$ is a unital positive linear map between unital C^* -algebras of operators acting on a Hilbert space \mathcal{H} , $A, B \in \mathcal{A}$ such that $\Phi(A^*A)$ or $\Phi(B^*B)$ is a contraction. Then*

$$(1 - \langle \Phi(B^*A)\xi, \xi \rangle)^2 \geq (1 - \langle \Phi(A^*A)\xi, \xi \rangle)(1 - \langle \Phi(B^*B)\xi, \xi \rangle)$$

for all unit vectors $\xi \in \mathcal{H}$.

Proof. Without loss of generality, assume that $\Phi(A^*A)$ is a contraction. Let $\xi \in \mathcal{H}$ be a unit vector. Hence $\langle \Phi(A^*A)\xi, \xi \rangle \leq 1$. Let us consider the quadratic polynomial

$$P(t) = (1 - \langle \Phi(A^*A)\xi, \xi \rangle)t^2 - 2(1 - \langle \Phi(B^*A)\xi, \xi \rangle)t + (1 - \langle \Phi(B^*B)\xi, \xi \rangle),$$

where $t \in \mathbb{R}$. It follows from the Cauchy–Schwarz inequality applied to the sesquilinear form $\langle A, B \rangle = \langle \Phi(B^*A)\xi, \xi \rangle$ that

$$P(1) = -\langle \Phi(A^*A)\xi, \xi \rangle + 2\langle \Phi(B^*A)\xi, \xi \rangle - \langle \Phi(B^*B)\xi, \xi \rangle \leq 0.$$

Clearly, $\lim_{t \rightarrow \infty} P(t) = \infty$. Hence the equation $P(t) = 0$ has a root in \mathbb{R} . Thus

$$(1 - \langle \Phi(B^*A)\xi, \xi \rangle)^2 - (1 - \langle \Phi(A^*A)\xi, \xi \rangle)(1 - \langle \Phi(B^*B)\xi, \xi \rangle) \geq 0$$

as desired. \square

The next result is immediately deduced from Theorem 3.1.

Corollary 3.3. *Let ψ be a positive linear functional on a C^* -algebra \mathcal{A} , let $A, B \in \mathcal{A}$ such that $\psi(A^*A) \leq M_1^2$ or $\psi(B^*B) \leq M_2^2$ for some positive numbers M_1, M_2 and let $L : \mathbb{C} \rightarrow \mathbb{R}$ be a function fulfilling $L(z) \leq |z|$ for all $z \in \mathbb{C}$. Then*

$$(M_1 M_2 - L(\psi(B^*A)))^2 \geq (M_1^2 - \psi(A^*A))(M_2^2 - \psi(B^*B)). \quad (3.1)$$

Corollary 3.4 (Aczél's Inequality). *If a_i, b_i ($1 \leq i \leq n$) are positive numbers such that $\sum_{i=2}^n a_i^2 < 1$ or $\sum_{i=2}^n b_i^2 < 1$, then*

$$\left(1 - \sum_{i=1}^n a_i b_i\right)^2 \geq \left(1 - \sum_{i=1}^n a_i^2\right) \left(1 - \sum_{i=1}^n b_i^2\right).$$

Apply Corollary 3.3 to the $n \times n$ matrices $A = \begin{bmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{bmatrix}$ and $B = \begin{bmatrix} b_1 & & 0 \\ & \ddots & \\ 0 & & b_n \end{bmatrix}$, positive linear functional $\text{tr}(\cdot)$ on $M_n(\mathbb{C})$ and $L(z) = |z|$.

If we assume that A and B are normal contractions of a unital C^* -algebra \mathcal{A} , $AB = BA$ and consider the positive sesquilinear form $\phi(C, D) = \psi(D^*C)$ ($C, D \in \mathcal{A}$), where ψ is a pure state (or, equivalently, a non-zero complex homomorphism) on the commutative C^* -algebra generated by three elements A, B and the identity I of \mathcal{A} , then we get from (3.1) that

$$(1 - \text{Re}(\psi(B^*A)))^2 \geq (1 - \psi(A^*A))(1 - \psi(B^*B)),$$

in which Re denotes the real part. Hence

$$\psi(1 - \text{Re}(B^*A))^2 \geq \psi((1 - A^*A)(1 - B^*B)),$$

whence

$$(1 - \text{Re}(B^*A))^2 \geq (1 - A^*A)(1 - B^*B).$$

The same assertion is valid with the imaginary part Im instead of Re . We proved therefore the following result.

Corollary 3.5. *Let A, B be commuting normal contractions of a unital C^* -algebra \mathcal{A} . Then*

$$(1 - \text{Re}(B^*A))^2 \geq (1 - A^*A)(1 - B^*B)$$

and

$$(1 - \text{Im}(B^*A))^2 \geq (1 - A^*A)(1 - B^*B).$$

Corollary 3.6. *Let ψ be a positive linear functional on $M_n(\mathbb{C})$, let $A, B \in M_n(\mathbb{C})$ such that $\psi(A) \leq M_1^2$ or $\psi(B) \leq M_2^2$ for some positive numbers M_1, M_2 . Then*

$$(M_1M_2 - \psi(A\sharp B))^2 \geq (M_1^2 - \psi(A))(M_2^2 - \psi(B)).$$

Proof. We may assume that $\psi(A) \leq M_1^2$ and $\psi(B) \leq M_2^2$. The positive linear functional ψ on $M_n(\mathbb{C})$ can be characterized by $\psi(C) = \langle CZ, Z \rangle$, where $Z \geq 0$ and $\langle \cdot, \cdot \rangle$ denotes the canonical inner product on $M_n(\mathbb{C})$ defined by $\langle X, Y \rangle = \text{tr}(Y^*X)$. It follows from (3.1) with $L(z) = |z|$ and elements $A^{1/2}$ and $(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}$ that

$$\begin{aligned} & (M_1M_2 - \langle (A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}Z, A^{1/2}Z \rangle)^2 \\ & \geq (M_1^2 - \langle (A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}Z, (A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}Z \rangle) \\ & \quad \times (M_2^2 - \langle A^{1/2}Z, A^{1/2}Z \rangle), \end{aligned}$$

or equivalently

$$\begin{aligned} & (M_1M_2 - \langle A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}Z, Z \rangle)^2 \\ & \geq (M_1^2 - \langle A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}Z, Z \rangle) \\ & \quad \times (M_2^2 - \langle A^{1/2}A^{1/2}Z, Z \rangle), \end{aligned}$$

whence

$$(M_1M_2 - \psi(A\sharp B))^2 \geq (M_1^2 - \psi(A))(M_2^2 - \psi(B)).$$

Note that $0 \leq \psi(A\sharp B) = |\psi(A\sharp B)|$. □

Finally, we present an Aczél type inequality involving unitarily invariant norms. Note that

$$|||AXB||| \leq \|A\| |||X||| \|B\| \tag{3.2}$$

for all X, A, B . The arithmetic-geometric mean inequality states that

$$|||A^*XB||| \leq \frac{1}{2}|||AA^*X + XBB^*|||. \quad (3.3)$$

Proposition 3.7. *Let $||| \cdot |||$ be a unitarily invariant norm on $M_n(\mathbb{C})$ and let $X, A, B \in M_n(\mathbb{C})$ such that $\|A\|^2 |||X||| \leq 1$ or $\|B\|^2 |||X||| \leq 1$. Then*

$$(1 - |||A^*XB|||)^2 \geq (1 - \|A\|^2 |||X|||)(1 - \|B\|^2 |||X|||)$$

Proof.

$$\begin{aligned} (1 - |||A^*XB|||)^2 &\geq \left(1 - \frac{1}{2}|||AA^*X + XBB^*|||\right)^2 && \text{(by (3.3))} \\ &\geq \frac{(1 - |||AA^*X|||) + (1 - |||XBB^*|||)}{2} && \text{(triangle Ineq.)} \\ &\geq (1 - |||AA^*X|||)(1 - |||XBB^*|||) && \text{(arithmetic-geometric mean ineq.)} \\ &\geq (1 - \|A\|^2 |||X|||)(1 - \|B\|^2 |||X|||) && \text{(by (3.2)).} \end{aligned}$$

□

REFERENCES

1. R. Bhatia, *Matrix Analysis*, Springer-Verlag, New York, 1997.
2. R. Bhatia and C. Davis, *More matrix forms of the arithmeticgeometric mean inequality*, SIAM J. Matrix Anal. **14** (1993) 132-136.
3. J. Aczél, *Some general methods in the theory of functional equations in one variable*, New applications of functional equations. (Russian) Uspehi Mat. Nauk (N.S.) **11** (1956), no. 3(69), 3-68.
4. Y.J. Cho, M. Matić and J.E. Pečarić, *Improvements of some inequalities of Aczél's type*, J. Math. Anal. Appl. **259** (2001), no. 1, 226-240.
5. S.S. Dragomir, *A generalization of J. Aczl's inequality in inner product spaces*, Acta Math. Hungar. **65** (1994), no. 2, 141-148.
6. S.S. Dragomir and B. Mond, *Some inequalities of Aczl type for gramians in inner product spaces*, Nonlinear Funct. Anal. Appl. **6** (2001), pp. 411-424.
7. J.I. Fujii, *Operator inequalities for Schwarz and Hua*, Sci. Math. **2** (1999), no. 3, 263-268.
8. T. Furuta, J. Mićić Hot, J.E. Pečarić and Y. Seo, *Mond-Pečarić Method in Operator Inequalities*, Element, Zagreb, 2005.
9. A.M. Mercer, *Extensions of popovicius inequality using a general method*, J. Inequal. Pure Appl. Math. **4** (1) (2003) Article 11.
10. M.S. Moslehian, *Operator extensions of Hua's inequality*, Linear Algebra Appl. **430** (2009), no. 4, 1131-1139.
11. J.G. Murphy, *C*-Algebras and Operator Theory*, Academic Press, San Diego, 1990.
12. T. Popoviciu, *On an inequality*, Gaz. Mat. Fiz. Ser. A **11** (64) (1959), 451-461.

13. X.H. Sun, *Aczél–Chebyshev type inequality for positive linear functions*, J. Math. Anal. Appl. **245** (2000), 393–403.
14. S. Wu and L. Debnath, *A new generalization of Aczél’s inequality and its applications to an improvement of Bellman’s inequality*, Appl. Math. Lett. **21** (2008), no. 6, 588–593.
15. S. Wu, *Some improvements of Aczél’s inequality and Popoviciu’s inequality*, Comput. Math. Appl. **56** (2008), no. 5, 1196–1205.

DEPARTMENT OF PURE MATHEMATICS, CENTER OF EXCELLENCE IN ANALYSIS ON ALGEBRAIC STRUCTURES (CEAAS), FERDOWSI UNIVERSITY OF MASHHAD, P. O. Box 1159, MASHHAD 91775, IRAN.

E-mail address: moslehian@ferdowsi.um.ac.ir and moslehian@ams.org

URL: <http://profsite.um.ac.ir/~moslehian/>